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THE VARIATIONAL METHOD FOR THE SOLUTION OF THE COMBINED HEAT AND MASS TRANSFER PROBLEMS

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Abstract—The Kantorovich variational method has been used to obtain the approximate solutions of the nonlinear combined heat and mass transfer problems under first- and second-kind boundary conditions in an infinite plate. The solutions obtained have the property of convergence and the accuracy suitable for practical applications.

NOMENCLATURE

Fo	Fourier number (nondimensional time)
Ki_m	mass transfer Kirpichyov number
Ki_q	heat transfer Kirpichyov number
Ko^*	modified Kossovich number
L	Lagrangian depending on functions of one variable, Fo
$\mathscr{L}$	Lagrangian depending on functions of two variables, X and Fo
Lu	Luikov number
Pn	Posnov number
X	nondimensional space coordinate.

Greek symbols

$\theta_1(X, Fo)$ nondimensional temperature
$\theta_2(X, Fo)$ nondimensional mass transfer potential.

Superscripts

$\partial/\partial Fo$.

INTRODUCTION

IN RECENT times the variational method is finding increasing use in the analysis of diffusion, heat conduction, and also of combined transport of different substances [1–9]. When used for the solution of the above problems, the method has a number of advantages due to which it has been given growing attention. Indeed, the variational principles that underlie the method used in the present work allow one to formulate the problem more compactly since they involve both the fundamental equations and the boundary conditions of the problem. This approach enables one to apply direct variational methods for the solution of problems, such as the methods of Ritz, Trefftz, and Kantorovich *et al.* Then, the solutions obtained are characterized by an increasing, and sometimes appraisable, accuracy. Moreover, the stationary nature of the variable integral makes the variational method less sensitive to the errors in the functions which approximate the solution of the problem, as compared with the other methods. At times this allows one, already at the first stage of calculations, to obtain a good approximation to the exact solution without constructing a full solution. This property of the variational methods makes them especially attractive for the investigation of nonlinear problems when the construction of the full solution is found to be difficult or impossible. This is also favoured by the fact that the variational methods imply the possibility of using a variety of additional information on the problem, the physical included.

This paper considers the specifics of the Kantorovich variational method as applied to the solution of the nonlinear problems of combined heat and mass transfer. Traditionally, these problems are formulated in the form of the systems of partial differential equations which incorporate contributions from the direct and coupled effects of transfer. Such systems include the second derivatives of the unknown functions with respect to spatial coordinates and the first derivatives with respect to time, i.e. contain both the diffusion and dissipation terms.

For the first time the system of equations of this type was suggested by Luikov with reference to the phenomenon of drying [10]. Subsequently, it turned out that a similar technique can be used to formulate the mathematical models of a wide class of combined transfer phenomena in continuous media [11]. The current interest in the investigation of the nonlinear models of combined heat and mass transfer is spurred by the fact that the studies of the phenomena in other, far from physics, areas of research result in models of this kind. For example, in biology this is the space–time dynamics of biogeocenoses [12]; in sea ecology, the distribution of phytoplankton, zooplankton and nutrient substances over the depth from the water surface and with time [13]; in medicine, it is the spread of epidemic diseases in space and time [14], the growth of malignant tumours [15], etc. In spite of the fact that the derivation of such equations is based on the account for the specific laws governing the behaviour of particular systems, the final form of the equations and boundary conditions is close to the system of heat and mass transfer equations suggested by Luikov.

FORMULATION OF THE PROBLEM

The effectiveness of the variational approach in the solution of the above problems will be considered in the example of combined heat and mass transfer in an infinite porous plate. This is a one-dimensional problem. The coordinate plane $X = 0$ is assumed to be coincident with the middle of the plate, while the dimensionless coordinate X varies within the range $-1 \leqslant X \leqslant 1$. Assuming the properties of the medium, in which heat and mass transfer occurs (thermal conductivity, mass conductivity, Soret coefficient), to be dependent on temperature and mass transfer potential, the system of heat and mass transfer equations [11] can be written in the form

$$\dot{\theta}_1 = \frac{\partial}{\partial X}\left[(M_1 + Ko^* \; Pn \; Lu \; K)\frac{\partial \theta_1}{\partial X}\right] - Ko^* \; Lu \frac{\partial}{\partial X}\left(M_2 \frac{\partial \theta_2}{\partial X}\right),$$
$$\dot{\theta}_2 = -Lu \; Pn \frac{\partial}{\partial X}\left(K \frac{\partial \theta_1}{\partial X}\right) + Lu \frac{\partial}{\partial X}\left(M_2 \frac{\partial \theta_2}{\partial X}\right), \quad (1)$$

where $M_1 = 1 + m_1\theta_1$; $M_2 = 1 + m_2\theta_2$; $K = 1 + k\theta_1$; and $m_1, m_2, k = \text{const}$.

The problem will be considered subject to the constant boundary conditions of the first and second kind.

Following the approach considered in detail in ref. [6], the variational formulation of the heat and mass transfer problem (1) under these boundary conditions may be represented in the form of the Hamilton type variational principle with carrying out the limit $\delta I = 0$. In this case the functional I has the form [Fo_1 and Fo_2 being the arbitrary time moments ($Fo_1 < Fo_2$)]

$$I = \int_{Fo_1}^{Fo_2} \int_{-1}^{1} \mathscr{L} \, \mathrm{d}X \, \mathrm{d}Fo, \quad (2)$$

while the Lagrange function $\mathscr{L}$ is written as

$$\mathscr{L} = \frac{\Delta}{2}\left\{\left[\dot{\theta}_1 - \frac{\partial}{\partial X}\left[(M_1 + Ko^* \; Pn \; Lu \; K)\frac{\partial \theta_1}{\partial X}\right] + Ko^* \; Lu \frac{\partial}{\partial X}\left(M_2 \frac{\partial \theta_2}{\partial X}\right)\right]^2 + \left[\dot{\theta}_2 + Lu \; Pn \frac{\partial}{\partial X}\left(K \frac{\partial \theta_1}{\partial X}\right) - Lu \frac{\partial}{\partial X} \times \left(M_2 \frac{\partial \theta_2}{\partial X}\right)\right]^2\right\} \exp \frac{Fo}{\Delta}. \quad (3)$$

The parameter Δ occurring in the above expression is allowed to go to zero on completion of the procedure of functional variation. This leads to equations (1) as a condition for the functional (2) to be stationary within the class of functions statisfying the boundary conditions. Assuming, in fact, that any natural boundary conditions are absent and taking θ_1 and θ_2 to be the generalized coordinates, the system of Euler–Lagrange equations for functional (2) can be written as

$$\frac{\partial \mathscr{L}}{\partial \theta_i} - \frac{\partial}{\partial Fo}\frac{\partial \mathscr{L}}{\partial \dot{\theta}_i} - \frac{\partial}{\partial X}\frac{\partial \mathscr{L}}{\partial(\partial \theta_i/\partial X)} + \frac{\partial^2}{\partial X^2}\frac{\partial \mathscr{L}}{\partial(\partial^2\theta_i/\partial X^2)} = 0 \quad (i = 1, 2). \quad (4)$$

Using the Lagrange function (3) in equation (4), and after dividing both sides of the equation by $\exp(Fo/\Delta)$, the following expressions are obtained

$$\dot{\theta}_1 - \frac{\partial}{\partial X}\left[(M_1 + Ko^* \; Pn \; Lu \; K)\frac{\partial \theta_1}{\partial X}\right] + Ko^* \; Lu \frac{\partial}{\partial X}\left(M_2 \frac{\partial \theta_2}{\partial X}\right) = \Delta \cdot G_1\left(X, Fo, \theta_i, \dot{\theta}_i, \frac{\partial \theta_i}{\partial X}, \ldots\right),$$
$$\dot{\theta}_2 + Lu \; Pn \frac{\partial}{\partial X}\left(K \frac{\partial \theta_1}{\partial X}\right) - Lu \frac{\partial}{\partial X}\left(M_2 \frac{\partial \theta_2}{\partial X}\right) = \Delta \cdot G_2\left(X, Fo, \theta_i, \dot{\theta}_i, \frac{\partial \theta_i}{\partial X}, \ldots\right) \quad (i = 1, 2).$$

Here the functions of G_i on the RHSs do not contain the parameter Δ. Thus, the limit transition $\Delta \to 0$ brings these expressions to the form of equations (1).

An important fact in the considered variational principle is the adherence to the order in which the variation and limit transition are carried out. This very order of operations should also be observed in realizing a particular variational method based on, the Hamiltonian principle which involves the limit transition.

THE KANTOROVICH VARIATIONAL METHOD

The Kantorovich variational method [16] pertains to direct methods of problem solution. By its importance for the solution of differential equations, the method holds an intermediate position among those giving exact solutions and the Ritz and Bubnov–Galerkin methods. The use of the Ritz variational method for the solution of the heat and mass transfer problems [9] consists in the replacement of the extremal problem for the functional by the problem of finding the extrema of the functions of many variables. This is achieved by introducing into the solution-approximating functions the indeterminate constants which are then chosen so that these functions would best satisfy the variational problem.

When the Kantorovich method is used, the distribution of the heat and mass transfer potential fields should be approximated in the class of functions which contain, as unknown parameters, the functions of one variable (e.g. of time). Of course, this class of functions encompasses as a subclass the approximating functions used in the Ritz method. This is the reason for the higher accuracy of the Kantorovich method. The other advantage of the method lies in the fact that here only a portion of the expression, which gives the solution, is chosen arbitrarily. The other part of the functions is determined based on the nature of the problem. In this case, if the functions approximating the transfer potentials, are given by the expressions of the form

$$\theta_i(X, Fo) = \sum_{n=0}^{\infty} \phi_{in}(Fo)W_{in}(X) \quad (i = 1,2), \qquad (5)$$

where $W_{in}(X)$ are the coordinate functions, and $\phi_{in}(Fo)$ the functions to be determined, then, upon integration over the spatial coordinate, equation (2) is reduced to

$$I = \int_{Fo_1}^{Fo_2} L(Fo, \phi_{1n}, \phi_{2n}, \dot{\phi}_{1n}, \dot{\phi}_{2n})\, dFo. \qquad (6)$$

The Lagrangian L in the functional (6) depends only on the functions of one variable, Fo. Therefore, the Euler–Lagrange equations for it have the form

$$\frac{\partial \mathscr{L}}{\partial \phi_{in}} - \frac{\partial}{\partial Fo}\frac{\partial \mathscr{L}}{\partial \dot{\phi}_{in}} = 0 \ (i = 1,2;\ n = 0,1,\ldots,\infty). \qquad (7)$$

In our case equations (7) contain also the parameter Δ. The operation $\Delta \to 0$ brings equations (7) to a system of ordinary differential equations to determine the unknown functions $\phi_{in}(Fo)$ in equations (5).

BOUNDARY CONDITIONS OF THE FIRST KIND

Assume the following boundary conditions

$$\theta_i(-1, Fo) = \theta_i(1, Fo) = 0, \qquad (8)$$

$$\theta_i(X, 0) = 1 \ (i = 1,2). \qquad (9)$$

The solution will be sought in the form

$$\theta_i(X, Fo) = \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cos \frac{(2n+1)\pi X}{2}\, \phi_{in}(Fo) \ (i = 1,2). \qquad (10)$$

The boundary conditions (8) are fulfilled for equations (10). If the functions ϕ_{in} to be determined are made to obey the condition

$$\phi_{in}(0) = 1 \ (i = 1,2;\ n = 0,1,\ldots,\infty), \qquad (11)$$

then the initial conditions (9) will also be fulfilled. In this case the compliance with the boundary conditions is ensured by the construction of the approximating functions (10) on the basis of the familiar exact solution of the heat conduction equation [17] obtained under the same boundary conditions.

Substitute equations (10) into the Lagrange function (3) and integrate over X in the functional (2). Then setting up the Euler–Lagrange equations in the form of equations (7) and carrying out the limit transition $\Delta \to 0$, the following system of differential equations for the ϕ_{in} functions is eventually arrived at

$$\begin{aligned}\dot{\phi}_{1p} = (2p+1)^2 \Big[& 8 \sum_{n=0}^{\infty} N(T\phi_{1n}^2 - m_2 A_2 \phi_{2n}^2) \\ & + \sum_{n,l=0}^{\infty} P(T\phi_{1n}\phi_{1l} - m_2 A_2 \phi_{2n}\phi_{2l}) \\ & - \sum_{n=0}^{\infty} Q(T\phi_{1n}\phi_{1p} - m_2 A_2 \phi_{2n}\phi_{2p}) \Big] \\ & - \frac{\pi^2}{4}(A_1\phi_{1p} - A_2\phi_{2p}), \\ \dot{\phi}_{2p} = (2p+1)^2 \Big[& 8 \sum_{n=0}^{\infty} N(m_2 A_3 \phi_{2n}^2 - kA_4\phi_{1n}^2) \\ & + \sum_{n,l=0}^{\infty} P(m_2 A_3 \phi_{2n}\phi_{2l} - kA_4\phi_{1n}\phi_{1l}) \\ & - \sum_{n=0}^{\infty} Q(m_2 A_3 \phi_{2n}\phi_{2p}) - kA_4\phi_{1n}\phi_{1p} \Big] \\ & - \frac{\pi^2}{4}(A_3\phi_{2p} - A_4\phi_{1p}) \ (p = 0,1,\ldots,\infty), \end{aligned} \qquad (12)$$

where

$A_1 = 1 + Ko^*\, Pn\, Lu$; $A_2 = Ko^*\, Lu$; $A_3 = Lu$;

$A_4 = Lu\, Pn$;

$T = m_1 + k\, Ko^*\, Pn\, Lu$; $N = \dfrac{\tau}{\lambda\mu\nu} - \dfrac{1}{3\tau^2}$;

$P = \dfrac{1}{\lambda}\left[\alpha + \beta + \gamma + \eta - \dfrac{\varepsilon}{\tau}(\alpha + \beta - \gamma - \eta)\right]$;

$Q = \left(\dfrac{1}{\tau} + \dfrac{\tau}{\lambda^2}\right)\left(\dfrac{2}{\tau} - \dfrac{1}{\mu} - \dfrac{1}{\nu}\right) - \dfrac{2\tau}{\lambda\mu\nu}$;

$\lambda = 2p+1$; $\mu = 4n - 2p + 1$; $\nu = 4n + 2p + 3$;

$\tau = 2n+1$; $\varepsilon = 2l+1$; $\alpha = [2(p-n-l)-1]^{-1}$;

$\beta = [2(p-n+l)+1]^{-1}$; $\gamma = [2(p+n+l)+3]^{-1}$;

$\eta = [2(p+n-l)+1]^{-1}$.

The above expressions constitute a system of ordinary first-order differential equations written down in normal form.

If in the initial system of equations (1) [and accordingly in the Lagrangian (3)] it is assumed that $m_1 = m_2 = k = 0$, then the nonlinear problem of combined heat and mass transfer is transformed into a linear problem with the boundary conditions (8) and (9). In this case, equations (12) take on the form

$$\dot{\phi}_{1p} = -\frac{\pi^2}{4}(2p+1)^2(A_1\phi_{1p} - A_2\phi_{2p}),$$

$$\dot{\phi}_{2p} = \frac{\pi^2}{4}(2p+1)^2(A_4\phi_{1p} - A_3\phi_{2p})$$

$$(p = 0,1,\ldots,\infty). \qquad (13)$$

The solution of the system of homogeneous differential equations (13), for example, at different real roots of the characteristic equation s_1 and s_2 and initial conditions (11) is written as

$$\phi_{1p}(Fo) = \frac{1-\lambda_2}{\lambda_1-\lambda_2} \exp s_1\, Fo - \frac{1-\lambda_1}{\lambda_1-\lambda_2} \exp s_2\, Fo,$$

$$\phi_{2p}(Fo) = \lambda_1 \frac{1-\lambda_2}{\lambda_1-\lambda_2} \exp s_1\, Fo$$

$$-\lambda_2 \frac{1-\lambda_1}{\lambda_1-\lambda_2} \exp s_2\, Fo \quad (p = 0, 1, \ldots, \infty),$$

where

$$\lambda_i = \frac{1}{A_2}\left[A_1 + \frac{4s_i}{\pi^2(2p+1)^2}\right] (i = 1, 2);$$

$$s_{1,2} = -\frac{\pi^2}{8}(2p+1)^2 \{A_1 + A_3$$

$$\pm\sqrt{[(A_1+A_3)^2 - 4(A_1A_3 - A_2A_4)]}\}.$$

If in equation (1) it is additionally assumed that $Ko^* = Pn = Lu = 0$, then the problem of combined heat and mass transfer degenerates into the linear heat conduction equation, while equation (13) takes on the form (the first subscript is omitted)

$$\dot{\phi}_p = -\frac{\pi^2}{4}(2p+1)^2\phi_p.$$

This equation, with account for the initial conditions, yields

$$\phi_p(Fo) = \exp\left[-\frac{\pi^2}{4}(2p+1)^2 Fo\right] (p = 0, 1, \ldots, \infty). \quad (14)$$

The substitution of equation (14) into equation (10) gives the following expression for the temperature distribution

$$\theta(X, Fo) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} \times \exp\left[-\frac{\pi^2}{4}(2n+1)^2 Fo\right]\cos\frac{(2n+1)\pi X}{2},$$

which completely coincides with the familiar solution of the linear heat conduction problem under the considered boundary conditions obtained by another method [17].

In the general case the solution of the system of nonlinear equations (12) is unobtainable in analytical form. However, at specific coefficients its solution is easily obtained by numerical methods. For example, the use of the Runge–Kutta method at $m_1 = m_2 = k = 0.1$ and 0.5 and similarity numbers $Ko^* = 0.33$, $Pn = 0.5$ and $Lu = 2.0$ gives the following table (Table 1) of results for the first three terms of each series (10).

Figure 1 shows the distribution of the heat and mass transfer potentials obtained by the Kantorovich method at the same similarity numbers and coefficients $m_1 = m_2 = k = 0.1$. It also contains the results of the solution of this problem by the Ritz method [9] and by the finite-difference method with application of the explicit difference scheme. The comparison shows a satisfactory agreement between the results obtained by different methods. If, in this case, the solution given by the finite-difference method is taken as a standard one, then it can be noted that the Kantorovich method entails lesser errors than the Ritz method.

The examination of equation (10) at different values of the coefficients in equation (1) allows one to trace the convergence of approximate solutions for θ_1 and θ_2 on an increase in the number of series terms. For example, for the coefficients m_1, m_2, and k in the range from 0 to 1 the inclusion of the second terms in the functions (10) alters the solution, obtained only with the first term, by no more than 2.8%. The inclusion of the third term of the series alters the solution by less than 0.4%.

BOUNDARY CONDITIONS OF THE SECOND KIND

Consider the system of equations (1) with the boundary conditions of the form

$$\frac{\partial\theta_i}{\partial X}(1, Fo) = Q_i, \frac{\partial\theta_i}{\partial X}(-1, Fo) = -Q_i,$$

$$\theta_i(X, 0) = 0\ (i = 1, 2) \quad (15)$$

Table 1. The values of the first three functions ϕ_{ij} in equations (10) at $m_1 = m_2 = k = 0.1$ and 0.5 ($Ko^* = 0.33$, $Pn = 0.5$, $Lu = 2.0$)

		ϕ_{ij}					
Fo	m_1, m_2, k	ϕ_{10}	ϕ_{11}	ϕ_{12}	ϕ_{20}	ϕ_{21}	ϕ_{22}
0.2	0.1	0.7246	−0.0499	0.0709	0.6491	−0.0255	0.1416
	0.5	0.8086	−0.5499	0.0745	0.7323	−0.3822	0.1309
0.4	0.1	0.5102	−0.0400	0.0708	0.4337	−0.0289	0.1412
	0.5	0.6281	−0.5728	0.0734	0.5253	−0.3551	0.1295
0.6	0.1	0.3537	−0.0196	0.0706	0.2934	−0.0138	0.1410
	0.5	0.4692	−0.2982	0.0707	0.3755	−0.1829	0.1315
0.8	0.1	0.2431	−0.0091	0.0706	0.2003	−0.0063	0.1409
	0.5	0.3397	−0.1319	0.0695	0.2681	−0.0816	0.1337
1.0	0.1	0.1664	−0.0042	0.0706	0.1367	−0.0029	0.1408
	0.5	0.2407	−0.0572	0.0679	0.1904	−0.0361	0.1341

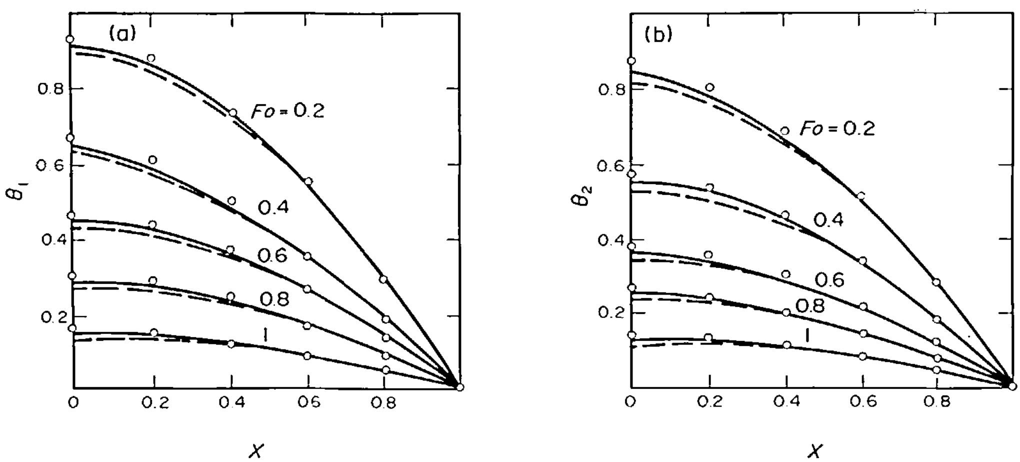

FIG. 1. The distribution of the fields of nondimensional transport parameters at $X \geqslant 0$ and first-kind boundary conditions: (a) for θ_1; (b) for θ_2 (——, the solution by the Kantorovich method; ---, by the finite-difference method; O, by the Ritz method). $Ko^* = 0.33$, $Pn = 0.50$, $Lu = 2.0$; $m_1 = m_2 = k = 0.1$.

where $Q_1 = Ki_q$; $Q_2 = Pn\ Ki_q + Ki_m$; Ki_q, Ki_m = const.

The approximating functions which satisfy the boundary conditions (15) can be constructed using the structure of the familiar solution of the linear heat and mass transfer problem subject to the second-kind boundary condition [11].

Let us assume that

$$\theta_1(X, Fo) = (Ki_q - Ko^*\ Lu\ Ki_m)Fo - \frac{1}{6}(1-3X^2)Ki_q - \sum_{n=1}^{\infty}\sum_{i=1}^{2}(-1)^n\frac{2}{(n\cdot\pi)^2}C_i^q \times \cos(n\pi X)\phi_{1ni}(Fo),$$

$$\theta_2(X, Fo) = Ki_m\ Lu\ Fo - \frac{1}{6}(1-3X^2)(Pn\ Ki_q + Ki_m) - \sum_{n=1}^{\infty}\sum_{i=1}^{2}(-1)^n\frac{2}{(n\pi)^2}C_i^m \times \cos(n\pi X)\phi_{2ni}(Fo),$$

where

$$C_j^q = (-1)^{3-j}\frac{Ki_q(v_{3-j}^2 - 1/Lu) + Ko^*\ Ki_m}{v_2^2 - v_1^2};$$

$$C_j^m = (-1)^{3-j}\frac{Pn\ Ki_q\ v_{3-j}^2 - (1-v_{3-j})Ki_m}{v_2^2 - v_1^2};$$

$$v_j^2 = \frac{1}{2}\left\{\left(1 + Ko^*\ Pn + \frac{1}{Lu}\right) + (-1)^j \times \sqrt{\left[\left(1 + Ko^*\ Pn + \frac{1}{Lu}\right)^2 + \frac{4}{Lu}\right]}\right\}\ (j = 1,2). \quad (16)$$

In the exact solution of the linear heat and mass transfer problem [11] the functions ϕ_{1ni} and ϕ_{2ni} are defined and have the form

$$\phi_{1ni} = \exp\left[-(\pi n)^2 v_i^2\ Lu\ Fo\right],$$

$$\phi_{2ni} = \exp\left[-(\pi n)^2 v_i^2\ Lu\ Fo\right].$$

Therefore, when determining the functions ϕ_{1ni} and ϕ_{2ni}, which are unknown in our case, the initial conditions should have the form

$$\phi_{1ni}(0) = \phi_{2ni}(0) = 1. \quad (17)$$

Substituting equation (16) into (3) and performing the same operations as those used for the solution of a similar problem with the boundary conditions of the first kind, the following system of equations is obtained to determine the unknown functions ϕ_{1ni} and ϕ_{2ni} in equations (16)

$$\dot{\phi}_{111} = -\left[\pi^2(A_1 + RT\ Fo) - \frac{7}{4}Ki_q\ T\right]\phi_{111} - \frac{8}{9}Ki_q\ T\phi_{121} + A_2B\left[\pi^2(1 + A_3 m_2\ Ki_m\ Fo) - \frac{7}{4}m_2 Q_2\right]\phi_{211} + \frac{8}{9}A_2 Q_2 m_2 B\phi_{221} + \frac{1}{4}T(C_1^q + C_2^q)\phi_{111}\phi_{121} - \frac{1}{4}F\phi_{211}\phi_{221} - D,$$

$$\dot{\phi}_{121} = \frac{176}{9}Ki_q\ T\phi_{111} - \left[4\pi^2(A_1 + RT\ Fo) - \frac{9}{4}Ki_q T\right]\phi_{121} - \frac{176}{9}A_2 Q_2 m_2 B\phi_{211} + A_2 B\left[4\pi^2(1 + A_3 m_2\ Ki_m\ Fo) - \frac{9}{4}m_2 Q_2\right]\phi_{221} + 8T(C_1^q + C_2^q)\phi_{111}^2 - 8F\phi_{211}^2 - D,$$

$$\dot{\phi}_{211} = A_4 B\left[(\pi^2(1+kR\ Fo) - \frac{7}{4}k\ Ki_q)\right]\phi_{111} + \frac{8}{9}A_4 k\ Ki_q\ B\phi_{121} - A_3\left[\pi^2(1+A_3 m_2\ Ki_m\ Fo) - \frac{7}{4}m_2 Q_2\right]\phi_{211} - \frac{8}{9}A_3 m_2 Q_2 \phi_{221} - \frac{1}{4}G\phi_{111}\phi_{121} + \frac{1}{4}V\phi_{211}\phi_{221} + E,$$

$$\dot{\phi}_{221} = -\frac{176}{9}A_4 k\ Ki_q\ B\phi_{111} + A_4 B\left[4\pi^2(1+kR\ Fo) - \frac{9}{4}k\ Ki_q\right]\phi_{121} + \frac{176}{9}A_3 Q_2 m_2 \phi_{211} - A_3\left[4\pi^2(1+A_3 m_2\ Ki_m\ Fo) - \frac{9}{4}m_2 Q_2\right]\phi_{221} - 8G\phi_{111}^2 + 8V\phi_{211}^2 + E,$$

$$\phi_{112} = \phi_{111},\ \phi_{122} = \phi_{121},\quad \phi_{212} = \phi_{211},\ \phi_{222} = \phi_{221}, \qquad (18)$$

where

$$R = Ki_q - Ko^*\ Lu\ Ki_m;\ B = \frac{C_1^m + C_2^m}{C_1^q + C_2^q};$$

$$D = \frac{3}{C_1^q + C_2^q}(T\ Ki_q^2 - A_2 m_2 Q_2^2);$$

$$F = A_2 m_2 \frac{(C_1^m + C_2^m)^2}{C_1^q + C_2^q};$$

$$E = \frac{3}{C_1^m + C_2^m}(A_4 k\ Ki_q^2 - A_3 m_2 Q_2^2);$$

$$G = A_4 k \frac{(C_1^q + C_2^q)^2}{C_1^m + C_2^m};$$

$$V = A_3 m_2 (C_1^m + C_2^m).$$

The system of equations (18) includes four first-order differential equations that can be easily solved by numerical methods at the initial conditions (17), while at $m_1 = m_2 = k = 0$ (the linear heat and mass transfer problem) can also be solved in analytical form.

Table 2 contains, as an example, the results of computer calculations of these functions at $m_1 = m_2 = k = 0$ and $m_1 = m_2 = k = 0.5$ (nonlinear problem). The following values of the similarity numbers were used: $Ko^* = 1.4$, $Pn = 0.5$, $Lu = 0.6$, $Ki_q = 1$, and $Ki_m = 0.5$.

Figure 2 gives the comparison of the approximate solutions of the linear heat and mass transfer problem at the second-kind boundary conditions with the exact solutions of this problem [11]. The comparison shows a good coincidence of the results, while the limiting absolute error of the approximate computer solutions is calculated to be 1.8% for θ_1 and 2.1% for θ_2. In both cases it is attained at $X = 1.0$ and small values of Fo.

Figure 3 presents the solutions of the nonlinear heat and mass transfer problem for θ_1 and θ_2. The comparison with the solutions of the linear problem (Fig. 2) reveals a certain retardation in the process of heat and mass transfer at the chosen values of coefficients and similarity numbers in the nonlinear case.

Just as under the first-kind boundary conditions, the basic contribution to the solution of the problem is made by the first terms of the approximating expressions (16). In fact, the inclusion of the second term in the series (16) alters the solution by less than 1.5%. A good accuracy and convergence of the variational solutions at the second-kind boundary conditions can be explained by an advantageous choice of the approximating functions on the basis of the solutions for the linear heat and mass transfer problem.

CONCLUSION

The Hamilton variational principle with the limit transition and the Kantorovich method allow the solution of the problems of combined heat and mass transfer in porous bodies when the thermophysical elements depend on the transfer potentials. The boundary-value problem for the system of partial differential equations is replaced in this case by the Cauchy problem for the system of ordinary differential equations. These are derived in a well-substantiated form: have the first order, normal form, constant coefficients and the initial conditions convenient for numerical calculation. In the case of a linear heat and mass transfer, the equations obtained are solved analytically, while in the nonlinear case, by the well known and tested numerical methods on a computer. This would require machine time 3–4 times less than for the full solution of the problem by the finite-difference method.

The choice of the approximating functions based on the solution of the linear problems of heat conduction and combined heat and mass transfer with the same boundary conditions allows one to obtain rather accurate and convergable distributions of the potentials of heat and mass transfer in a bound substance. A satisfactory accuracy is achieved already after the use of the first three or four terms of the series. The use of such approximating functions expands the range of applicability of the well-known solutions to the linear problems of transfer and increases their practical significance.

The solutions obtained in the present work for the nonlinear problems of combined heat and mass transfer under the first- and second-kind boundary conditions allow one to trace the effect of the nonlinear characteristics of the conducting medium on the

Table 2. The values of the functions ϕ_{ijl} in equations (16) at $m_1 = m_2 = k = 0$ and 0.5 ($Ko^* = 1.4$, $Pn = 0.5$, $Lu = 0.6$, $Ki_q = 1$, $Ki_m = 0.5$)

		ϕ_{ijl}			
Fo	m_1, m_2, k	ϕ_{111}	ϕ_{121}	ϕ_{211}	ϕ_{221}
0.2	0.0	0.4127	0.0477	0.5114	0.0602
	0.5	0.3305	0.1152	0.4563	0.1710
0.4	0.0	0.1991	0.0027	0.2512	0.0034
	0.5	0.0814	0.0034	0.1507	0.0271
0.6	0.0	0.0974	0.0002	0.1230	0.0002
	0.5	−0.0338	−0.0362	−0.0042	−0.0331
0.8	0.0	0.0477	0.0000	0.0602	0.0000
	0.5	−0.0855	−0.0546	−0.0791	−0.0652
1.0	0.0	0.0233	0.0000	0.0295	0.0000
	0.5	−0.1065	−0.0628	−0.1130	−0.0826

$\phi_{112} = \phi_{111}, \phi_{122} = \phi_{121}, \phi_{212} = \phi_{211}, \phi_{222} = \phi_{221}.$

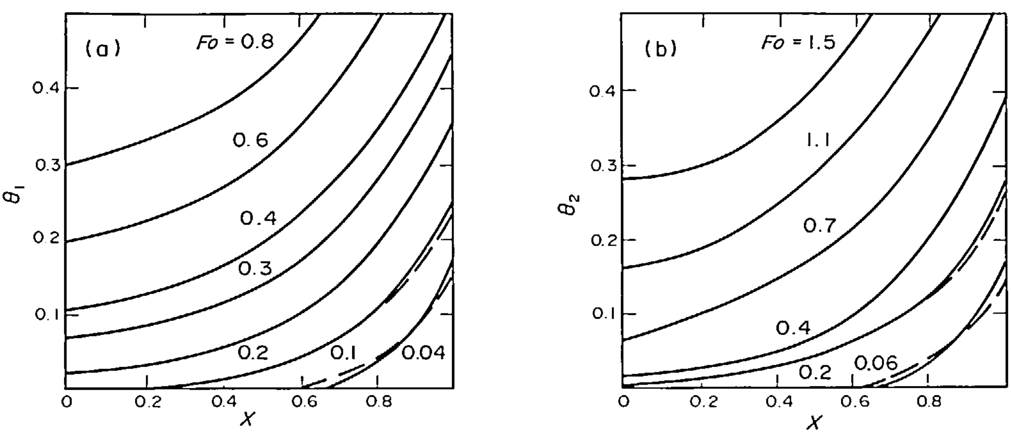

FIG. 2. The comparison between the approximate solutions of the linear problem of combined heat and mass transfer obtained by the Kantorovich method and the exact solutions at the second-kind boundary conditions: (a) for θ_1; (b) for θ_2 (——, approximate solutions; – –, exact solutions). $Ko^* = 1.4$, $Pn = 0.5$, $Lu = 0.6$, $Ki_q = 1$, $Ki_m = 0.5$.

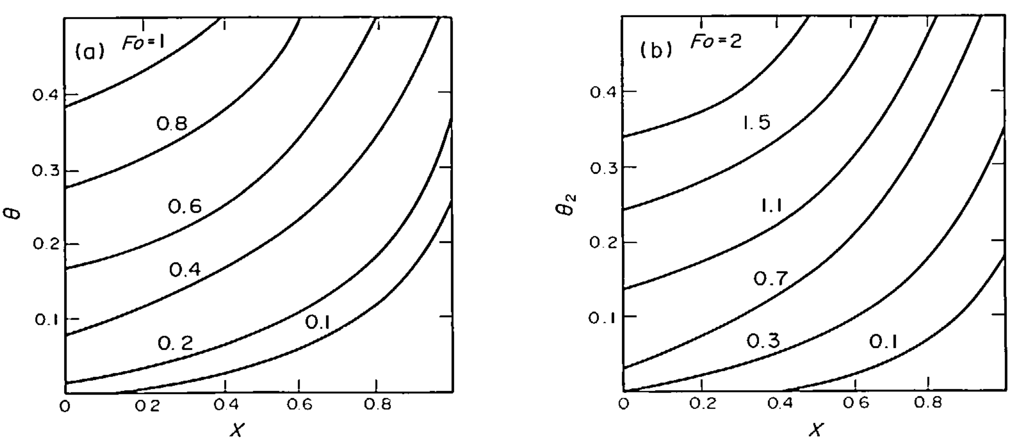

FIG. 3. The distribution of the fields of nondimensional transport parameters at $X \geqslant 0$ and second-kind boundary conditions: (a) for ϕ_1; (b) for θ_2. $Ko^* = 1.4$, $Pn = 0.5$, $Lu = 0.6$, $Ki_q = 1$, $Ki_m = 0.5$; $m_1 = m_2 = k = 0.5$.

process of combined transfer. The investigation of equations (10) and (16) for the coefficients m_1, m_2, and k varying in the functions M_1, M_2 and K from 0 to 1 shows that a simultaneous increase of these coefficients leads to a slight retardation of the heat and mass transfer process. The same values of θ_1 and θ_2 at a fixed point in the medium on an increase of m_1, m_2 and k are obtained, in this case, for a larger time interval.

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METHODE VARIATIONNELLE POUR LA SOLUTION DES PROBLEMES DE TRANSFERTS COMBINES DE CHALEUR ET DE MASSE

Résumé—La méthode variationnelle de Kantorovich est utilisée pour obtenir les solutions approchées des problèmes non linéaires de transferts combinés de chaleur et de masse avec des conditions aux limites de première et de seconde espèce dans une plaque infinie. Les solutions obtenues ont la propriété de convergence et la précision souhaitable pour des applications pratiques.

LÖSUNG DES KOMBINIERTEN WÄRME- UND STOFFÜBERGANGSPROBLEMS MIT HILFE DER VARIATIONSRECHNUNG

Zusammenfassung—Das Verfahren der Variationsrechnung nach Kantorovich wurde verwendet, um Näherungslösungen des nicht-linearen kombinierten Wärme- und Stoffübergangsproblems bei Randbedingungen erster und zweiter Art für eine unendliche Platte zu erhalten. Die gewonnenen Lösungen sind konvergent und genau genug für praktische Anwendungen.

ВАРИАЦИОННЫЙ МЕТОД РЕШЕНИЯ ЗАДАЧ ВЗАИМОСВЯЗАННОГО ТЕПЛО-И МАССОПЕРЕНОСА

Аннотация—Вариационный метод Канторовича использован для получения приближенных решений нелинейных задач взаимосвязанного тепло- и массопереноса при граничных условиях I и II рода в неограниченной пластине. Полученные решения задач обладают свойством сходимости и достаточной в инженерной практике точностью.